

Regular Separability of Well Structured Transition Systems

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Abstract. We investigate languages recognized by well structured transition systems (WSTS) with upward (resp. downward) compatibility. We show that under mild assumptions every two disjoint WSTS languages are regular separable, i.e., there exists a regular language containing one of them and disjoint from the other. In particular, if a language, as well as its complement, are both recognized by a WSTS, then they are necessarily regular.

1 Introduction

In this paper we study languages recognized by *well structured transition systems* (WSTS in short). WSTSes, introduced as an abstract framework by Finkel and Schnoebelen [14], are a widely investigated family of transition systems where the space of configurations is (typically) infinite but equipped with a well quasi order (wqo). Furthermore, the transition relation satisfies a compatibility condition with respect to the quasi order. This abstract setting subsumes many widely studied models, like vector addition systems (VAS) or lossy channel systems (LCS), and attracted recently a significant attention as it enjoys numerous general decidability results. Indeed, already in the late 80-ties it has been noticed by Finkel [10,11] that termination, boundedness and coverability problems are decidable for this kind of systems.

In this paper we concentrate on languages of finite words recognized by WSTSes. To this aim we consider *labelled* WSTSes with distinguished sets of initial and final configurations. We always assume the set of final configurations to be upward closed, or downward closed, with respect to the quasi order of a WSTS. The setting, when specialized to VAS with upward closed set of final configurations, subsumes so called *coverability languages* of VAS, where acceptance is by *covering* one of distinguished final configurations, not by *reaching* a final configuration.

Our contribution. We consider the regular separability problem for languages recognized by WSTSes. We say that a regular language S *separates* two languages K and L if S includes one of them and is disjoint with the other. The *regular separability problem* for a class \mathcal{G} of languages, asks whether two given languages $K, L \in \mathcal{G}$ are separable by some regular language.

This paper reports on a quite surprising observation: under mild assumptions, every two disjoint languages recognized by WSTSes are regular separable. In

particular, if a language, as well as its complement, are both recognized by a WSTS, then they both are necessarily regular!

Specifically, as our primary setting we consider WSTSes with *upward compatibility* (as defined in [14]), where the set of final configurations is upward closed. We use the name *upward-compatible WSTS* (UWSTS in short) in order to avoid confusion with a variant of WSTS admitting *downward compatibility*. Languages recognized by UWSTSes with finite branching we call *finitely-branching-UWSTS languages*. Finally, if both a language L and its complement are such languages, we call L *doubly finitely-branching-UWSTS language*. As our first result we observe that such languages are necessarily regular:

Theorem 1. *Every doubly finitely-branching-UWSTS language is regular.*

Our second result does not require the assumption of finite branching. On the other hand it requires the quasi order in an UWSTS to be not only a wqo, but even an ω^2 -wqo (a strengthening of the notion of wqo, where a natural lifting to powersets remains a wqo). We use the name ω^2 -UWSTS in this case. We observe that whenever the quasi order in an UWSTS is an ω^2 -wqo, regular separability holds trivially for every two disjoint languages:

Theorem 2. *Every two disjoint ω^2 -UWSTS languages are regular separable.*

Due to the ubiquity of WSTSes, applications of the results abound. Just to recall one case: every two disjoint coverability languages of VAS are regular separable, and if they are complements of each other then they are necessarily regular. And the same applies not only to other classes of WSTSes, for instance to languages of LCSes, but actually to *all* WSTS languages fulfilling the mild assumptions of Theorems 1 and 2. For instance, if the coverability language of a vector addition system is disjoint from the language of a lossy channel systems, they are necessarily regular separable.

In the proof of Theorem 2 we conveniently exploit a dual variant of WSTS, namely *downward-compatible WSTS* (DWSTS in short), where the set of final configurations is downward closed. It turns out that the reverse of every ω^2 -UWSTS is recognized by a deterministic DWSTS (cf. Lemma 5) and thus Theorem 2 follows easily from the following more fundamental result:

Theorem 3. *Every two disjoint deterministic DWSTS-languages are regular separable.*

(Note that the ω^2 -wqo assumption is no more needed here.)

Motivation and context. Separability is a classical problem in theoretical computer science. As a prominent example, the classical result says that every two co-recursively enumerable languages are recursively separable, i.e., separable by a recursive language [15]. In the area of formal languages, separability of regular languages by subclasses thereof was investigated most extensively. For the following separator subclasses, among the others, the separability problem of regular languages is decidable: the piecewise testable languages (shown independently by Czerwinski et al. [8], and by Place et al. [25]), the locally and locally

threshold testable languages (by Place et al. [24]), the languages definable in first order logic (by Place and Zeitoun [27]), and the languages of certain higher levels of the first order hierarchy (by Place and Zeitoun [26]).

Regular separability of classes larger than regular languages attracted little attention till very recently, as it typically yields an undecidable decision problem. As a remarkable example, already in 70's undecidability of regular separability of context-free languages has been shown by Szymanski and Williams [28] (cf. also a later proof by Hunt [17]); then the undecidability has been strengthened by Kopczyński to visibly pushdown languages [20] and by the authors of this paper to languages of one counter automata [7].

An intriguing problem, to the best of our knowledge remaining still open, is the decidability status of regular separability of languages of VAS (VAS languages), under a proviso that acceptance is by *reaching* a distinguished final configuration. As for now, positive answers are known only for some subclasses of VAS languages: PSPACE-completeness shown by us for one-counter nets (i.e., one-dimensional vector addition systems with states) [7], and elementary complexity shown by Clemente et al. for languages recognizable by Parikh automata (or, equivalently, by integer vector addition systems) [5]. Finally, regular separability of *commutative closures* of VAS languages has been shown decidable by Clemente et al. in [6]. As a consequence of this paper, regular separability of two VAS languages reduces to disjointness of the same two VAS languages (and is thus trivially decidable), under a proviso that acceptance is by *covering* a distinguished final configuration.

Languages of UWSTSes were investigated e.g. in [16], where many nice closure properties have been shown, including a natural pumping lemma. Various subclasses of languages of WSTSes were also considered in [9,1,23].

An interesting dichotomy that may be in some relation to the present paper has been shown recently in [21]: every family of languages closed under Boolean operations and rational transductions (Boolean closed full trio) is either included in the class of regular languages, or contains the whole arithmetical hierarchy.

Organisation. The rest of the paper is organized as follows. After preliminary Section 2, in Sections 3–4 we provide the proofs of Theorems 1–3. The last Section 5 contains final remarks.

2 Well structured transition systems

For a language $L \subseteq \Sigma^*$ and a word $w \in \Sigma^*$, the *left derivative* of w wrt. L is $w^{-1}L = \{v \in \Sigma^* \mid wv \in L\}$, and the *right derivative* of w wrt. L is $Lw^{-1} = \{v \in \Sigma^* \mid vw \in L\}$. The *reverse* of a word $w = a_1 \dots a_k \in \Sigma^*$ is $\text{rev}(w) = a_k \dots a_1$. The reverse of a language $L \subseteq \Sigma^*$ consists of reverses of all words in L , $\text{rev}(L) = \{\text{rev}(w) \mid w \in L\}$. Complement of a language L , i.e., the language $\Sigma^* \setminus L$, is denoted \bar{L} .

Well quasi orders. A quasi order (X, \preceq) , i.e. a set X equipped with a reflexive and transitive binary relation \preceq , is called *well quasi order* (wqo) if for every

infinite sequence $x_1, x_2, \dots \in X$ there exist indices $i < j$ such that $x_i \preceq x_j$. It is folklore that (X, \preceq) is wqo iff it admits neither an infinite descending sequence (i.e., is well-founded) nor an infinite antichain.

We will be working either with wqo, or with ω^2 -wqo, a strengthening of wqo. We prefer not to provide the standard definition of ω^2 -wqo (which can be found, e.g., in [22]), as it is technical and would not serve our aims in this paper. Instead, we take the characterization of Lemma 4 below as a working definition. The class of ω^2 -wqos provides a framework underlying the forward WSTS analysis developed in [12,13]. Both classes, namely wqos and ω^2 -wqos, are stable under various operations like Cartesian product; also the natural liftings of a wqo (resp. ω^2 -wqo) to finite multisets (multiset embedding) or finite sequences (Higman ordering) are wqos (resp. ω^2 -wqos) again.

A subset $U \subseteq X$ is *upward closed* with respect to \preceq if $u \in U$ and $u' \succeq u$ implies $u' \in U$. Similarly one defines *downward closed* sets. Clearly, U is upward closed if, and only if $X \setminus U$ is downward closed. The *upward* and *downward closure*, respectively, of a set $U \subseteq X$ is defined as:

$$\uparrow U = \{x \in X \mid \exists u \in U, x \succeq u\} \quad \text{and} \quad \downarrow U = \{x \in X \mid \exists u \in U, x \preceq u\}.$$

The family of all upward closed subset of X we denote by $\mathcal{P}^\uparrow(X)$, and the family of all downward closed subsets of X by $\mathcal{P}^\downarrow(X)$. If (X, \preceq) is a wqo then every upward closed set is the upward closure of a finite set, namely of the set of its minimal elements. This is not the case for downward closed set; we thus distinguish a subfamily $\mathcal{P}_{\text{fin}}^\downarrow(X) \subseteq \mathcal{P}^\downarrow(X)$ of downward closed subsets of X , containing downward closures of finite sets. In general these are not necessarily finite sets (consider the set $\mathbb{N} \cup \{\omega\}$ with ω bigger than all natural numbers, and the downward closure of $\{\omega\}$). The set $\mathcal{P}_{\text{fin}}^\downarrow(X)$, ordered by inclusion, is a wqo whenever (X, \preceq) is:

Claim. $(\mathcal{P}_{\text{fin}}^\downarrow(X), \subseteq)$ is a wqo, if (X, \preceq) is a wqo.

This property does not necessarily extend to the whole set $\mathcal{P}^\downarrow(X)$ of all downward closed subsets of X . As shown in [18]:

Lemma 4. $(\mathcal{P}^\downarrow(X), \subseteq)$ is a wqo if, and only if (X, \preceq) is an ω^2 -wqo.

(As a matter of fact, [18] considers the reverse inclusion order on upward closed sets, which is clearly isomorphic to the inclusion order on downward closed sets.)

Upward-compatible well structured transition systems. A labelled transition system (LTS) $\mathcal{S} = (S, T)$ over a finite alphabet Σ consists of a set of *configurations* S and a set of *transitions* $T \subseteq S \times \Sigma \times S$. We write $s \xrightarrow{a} s'$ instead of $(s, a, s') \in T$. A path from configuration s to configuration s' over a word $w = a_0 \dots a_{k-1}$ is a sequence of configurations $s = s_0, s_1, \dots, s_{k-1}, s_k = s'$ such that $s_i \xrightarrow{a_i} s_{i+1}$ for all $i \in \{0, \dots, k-1\}$. We write $s \xrightarrow{w} s'$. An LTS (S, T) is *finitely branching* if for every configuration $s \in S$ and $a \in \Sigma$ there is only finitely many configurations $s' \in S$ such that $s \xrightarrow{a} s'$.

Now we define a labeled version of *well structured transition systems* of [14]. An upward-compatible well structured transition system (UWSTS) $\mathcal{W} = (S, T, \preceq, I, F)$ over finite alphabet Σ is an LTS (S, T) over Σ equipped with two sets $I, F \subseteq S$ of initial and final configurations, respectively, and a well quasi order \preceq over configurations satisfying the following upward compatibility¹: whenever $s \preceq s'$ and $s \xrightarrow{a} r$, then $s' \xrightarrow{a} r'$ for some $r' \in S$ such that $r \preceq r'$. In other words, \preceq is a simulation relation. In this paper we assume that the set of initial configurations I of an UWSTS is finite, and the set of final configurations F is upward closed² with respect to \preceq .

\mathcal{W} is *deterministic* if it has exactly one initial configuration, and for every $s \in S$ and $a \in \Sigma$ there is exactly one $s' \in S$ such that $s \xrightarrow{a} s'$.

The language recognized by \mathcal{W} , denoted $L(\mathcal{W})$, is the set of words which occur on some path starting in an initial configuration and ending in a final one, i.e.

$$L(\mathcal{W}) = \{w \in \Sigma^* \mid \exists i \in I, f \in F, i \xrightarrow{w} f\}.$$

We observe that upward compatibility extends to words, which can be shown by simple induction on the length of the word:

Claim. Let $w \in \Sigma^*$. Whenever $s \preceq s'$ and $s \xrightarrow{w} r$, then $s' \xrightarrow{w} r'$ for some $r' \in S$ such that $r \preceq r'$.

As F is upward closed, w.l.o.g. we could replace the set I by its (not necessarily finite) downward closure $\downarrow I$ with respect to \preceq . Indeed, the replacement preserves the language of a WSTS: according to Claim 2, if $i \xrightarrow{w} f$ for some $i \preceq i' \in I$ and $f \in F$, then $i' \xrightarrow{w} f'$, for some $f' \succeq f$; therefore $f' \in F$, and $w \in L(\mathcal{W})$.

Languages of UWSTSes we call UWSTS-languages. When \preceq is ω^2 -wqo, the UWSTS \mathcal{W} is called ω^2 -UWSTS, and its language ω^2 -UWSTS-language. When (S, T) is finitely branching, the UWSTS \mathcal{W} is called *finitely-branching-UWSTS*, and its language *finitely-branching-UWSTS-language*. If both L and its complement \bar{L} are finitely-branching-UWSTS-languages, we call both of them *doubly finitely-branching-UWSTS-languages*.

Note that when defining the WSTS-languages we did not allow for ε -steps. Even if ε -steps can be easily eliminated by pre-composing and post-composing every transition $s \xrightarrow{a} s'$ with the reflexive-transitive closure of $\xrightarrow{\varepsilon}$, we need to emphasize that this transformation does not necessarily preserve finite branching.

Downward-compatible well structured transition systems. A downward-compatible well structured transition system (DWSTS) is defined like its upward-compatible counterpart, with two modifications. First, instead of upward compatibility, we require its symmetric variant, namely *downward compatibility*: whenever $s' \preceq s$ and $s \xrightarrow{a} r$, then $s' \xrightarrow{a} r'$ for some $r' \in S$ such that $r' \preceq r$. (In other

¹ This is strong compatibility, in terminology of [14].

² Languages defined by upwards closed sets of final configurations are called *coverability languages* in the literature.

words, the inverse of \preceq is a simulation relation.) Second, we assume the set of final configurations F to be downward closed, instead of being upward closed. Similarly as upward compatibility, downward compatibility extends to words:

Claim. Let $w \in \Sigma^*$. Whenever $s' \preceq s$ and $s \xrightarrow{w} r$, then $s' \xrightarrow{w} r'$ for some $r' \in S$ such that $r' \preceq r$.

The language $L(\mathcal{W})$ of an DWSTS \mathcal{W} is defined exactly as in case of UWSTS. As above, we also speak of finitely-branching-DWSTS languages, or ω^2 -DWSTS languages.

In the sequel we will build on an observation that every ω^2 -UWSTS-language is recognised, up to a reverse, by a DWSTS:

Lemma 5. *Let $\mathcal{W} = (S, T, \preceq, I, F)$ be an ω^2 -UWSTS. Then there is a deterministic DWSTS that recognises the reverse of $L(\mathcal{W})$.*

Proof. We define a DWSTS $\mathcal{W}' = (S', T', \preceq', I', F')$ as follows. Configurations are upward closed subsets of S , i.e. $S' = \mathcal{P}^\uparrow(S)$. There is one initial configuration, namely $I' = \{F\}$. An upward closed subset $U \in \mathcal{P}^\uparrow(S)$ is accepting, i.e. $U \in F'$, if $U \cap I \neq \emptyset$; in other words, if U contains some element of I . There is a transition (U, a, V) in T' if V is the pre-image of U along a -labeled transitions of \mathcal{W} :

$$V = \{s \in S \mid \exists u \in U, s \xrightarrow{a} u\}.$$

We order the set $S' = \mathcal{P}^\uparrow(S)$ by inverse inclusion: $U \preceq' V$ if and only if $U \supseteq V$. As (S, \preceq) is ω^2 -wqo, by Lemma 4 we know that (S', \preceq') is a wqo. The set of final configurations F' is downward closed; indeed, if $U \in F'$ and $U \subseteq V$ then necessarily $V \in F'$, as required. Finally, we verify the downward compatibility condition: if $U \subseteq V$, then for every $a \in \Sigma$, the pre-image of U along a is included in the pre-image of V along a .

It remains to check that $L(\mathcal{W}') = \text{rev}(L(\mathcal{W}))$. Indeed, for every $i \in I$ we have the following equivalence: $i \xrightarrow{w} f$ in \mathcal{W} for some $f \in F$ if, and only if $F \xrightarrow{\text{rev}(w)} U$ in \mathcal{W}' for some U such that $i \in U$. Therefore $w \in L(\mathcal{W})$ if, and only if $\text{rev}(w) \in L(\mathcal{W}')$.

Remark 6. In the similar vein one shows the claim opposite to Lemma 5: the reverse of the language of every ω^2 -DWSTS is recognised by a deterministic UWSTS.

Examples of WSTSes. Various well known and intensively investigated models of computation happen to be either an UWSTS or DWSTS. The list of natural classes of systems which are UWSTSes contains, among the others: vector addition systems (VAS) and their extensions (like vector addition systems with reset arcs, or with transfer arcs); lossy counter machines [3]; string rewriting systems based on context-free grammars; lossy communicating finite state machines (aka lossy channel systems, LCS) [4]; and many others. In the first two models listed above the configurations are ordered by multiset embedding,

while in the remaining two ones the configurations are ordered by subsequence (Higman) ordering. The natural examples of UWSTS, including all models listed above, are typically ω^2 -UWSTSes and, when considered without ε -transitions, finitely-branching. Thus both Theorems 1 and 3 apply to them.

DWSTSes are less popular. A natural source of examples is *gainy* models, like gainy counter system machines or gainy communicating finite state machines. For a more detailed overview, see e.g. [14] (page 31).

3 Regularity of doubly finitely-branching-UWSTS-languages

In this section we prove Theorem 1, after formulating and proving a couple of auxiliary facts. For convenience fix a UWSTS $\mathcal{W} = (S, T, \preceq, I, F)$ over an alphabet Σ , and let $L = L(\mathcal{W})$. For a word $w \in \Sigma^*$ we will consider the downward closure of the set of all configurations reachable from I by some path over w . Formally, we put

$$\delta_{\mathcal{W}}(w) = \downarrow\{s \in S \mid \exists i \in I, i \xrightarrow{w} s\}.$$

Lemma 7. *For every words $w, v \in \Sigma^*$, $\delta_{\mathcal{W}}(w) \subseteq \delta_{\mathcal{W}}(v)$ implies $w^{-1}L \subseteq v^{-1}L$.*

Proof. Suppose $\delta_{\mathcal{W}}(w) \subseteq \delta_{\mathcal{W}}(v)$ and let $u \in w^{-1}L$. Thus $wu \in L$, i.e.,

$$i \xrightarrow{w} s \xrightarrow{u} f$$

for some $i \in I$, $s \in S$ and $f \in F$. By the inclusion $\delta_{\mathcal{W}}(w) \subseteq \delta_{\mathcal{W}}(v)$ we deduce that $s \in \delta_{\mathcal{W}}(v)$, i.e., $i' \xrightarrow{v} s'$ for some $i' \in I$ and $s' \succeq s$. Now by Claim 2 we get a configuration $f' \succeq f$ such that

$$s' \xrightarrow{u} f'$$

Thus $i' \xrightarrow{vu} f'$ and, as F is upward closed, $f' \in F$, which implies $vu \in L$, as required.

Lemma 8. *If \mathcal{W} is finitely branching then for every word $w \in \Sigma^*$, the set $\delta_{\mathcal{W}}(w)$ is the downward closure of a finite set.*

Proof. Due to finite branching of \mathcal{W} , for every $i \in I$ and $w \in \Sigma^*$, the set of all configurations s' with $i \xrightarrow{w} s'$ is finite. As I is finite, the set $\delta_{\mathcal{W}}(w)$ is the downward closure of a finite union of finite sets.

Towards the proof of Theorem 1, suppose $L = L(\mathcal{W}_1)$ and $\bar{L} = L(\mathcal{W}_2)$ for two finitely-branching-UWSTSes $\mathcal{W}_1 = (S_1, T_1, \preceq_1, I_1, F_1)$ and $\mathcal{W}_2 = (S_2, T_2, \preceq_2, I_2, F_2)$. Recall the Myhill-Nerode equivalence: $w \sim_L v$ if, and only if $w^{-1}L = v^{-1}L$. Our aim is to prove that \sim_L has finite index.

Consider representatives w_1, w_2, \dots of all equivalence classes of \sim_L . By Lemma 8 all the sets $\delta_{\mathcal{W}_1}(w_i) \subseteq S_1$ and $\delta_{\mathcal{W}_2}(w_i) \subseteq S_2$ are the downward closures of finite

sets. Thus for every $i = 1, 2, \dots$ the pair $\delta_i = (\delta_{\mathcal{W}_1}(w_i), \delta_{\mathcal{W}_2}(w_i))$ belongs to $X = \mathcal{P}_{\text{fin}}^\downarrow(S_1) \times \mathcal{P}_{\text{fin}}^\downarrow(S_2)$. Ordering X by point-wise inclusion, denoted \subseteq_2 , yields a wqo:

Claim. (X, \subseteq_2) is a wqo.

Indeed, the claim follows by Claim 2 together with the fact that Cartesian product preserves wqo. We are going to argue that the pairs δ_i are incomparable with respect to \subseteq_2 :

Claim. For $i \neq j$, the pairs δ_i and δ_j are incomparable with respect to \subseteq_2 .

Proof. Indeed, suppose $\delta_i \subseteq_2 \delta_j$, i.e., $\delta_{\mathcal{W}_1}(w_i) \subseteq \delta_{\mathcal{W}_1}(w_j)$ and $\delta_{\mathcal{W}_2}(w_i) \subseteq \delta_{\mathcal{W}_2}(w_j)$. By double application of Lemma 7 this would yield

$$w_i^{-1}L \subseteq w_j^{-1}L \quad \text{and} \quad w_i^{-1}\bar{L} \subseteq w_j^{-1}\bar{L}.$$

As $w_i^{-1}L$ and $w_i^{-1}\bar{L}$ are mutual complements, and likewise for $w_j^{-1}L$ and $w_j^{-1}\bar{L}$, the above inclusions would imply the equality $w_i^{-1}L = w_j^{-1}L$, which is impossible as w_i and w_j are representatives of different equivalence classes of \sim_L .

By the last claim the set of all pairs δ_i is an antichain with respect to \subseteq_2 , and thus necessarily finite by Claim 3. This completes the proof of Theorem 1.

Remark 9. As a matter of fact, in the proof we did not need \subseteq_2 to be well-founded; we only used the fact that it does not admit infinite antichains.

4 Regular separability

In this section we prove Theorems 2 and 3. We start by observing that due to Lemma 5, Theorem 3 implies Theorem 2. Indeed, the reverses of two disjoint languages are still disjoint, and regular languages are closed under reverses. Comparing to the proof of Theorem 1, where we did a forward analysis of a UWSTS, proving Theorem 2 requires a backward analysis, as done in the proof of Lemma 5. This is the reason why we need the ω^2 -wqo assumption; on the other hand finite branching assumption is not needed any more.

In the rest of this section we concentrate on proving Theorem 3.

To this aim we fix two deterministic DWSTSes $\mathcal{W}_1 = (S_1, T_1, \preceq_1, \{i_1\}, F_1)$ and $\mathcal{W}_2 = (S_2, T_2, \preceq_2, \{i_2\}, F_2)$ over the same alphabet Σ , and assume that $L_1 = L(\mathcal{W}_1)$ and $L_2 = L(\mathcal{W}_2)$ are disjoint. We are going to construct an NFA $\mathcal{A} = (Q, T, q, F)$ over the alphabet Σ such that $L_1 \subseteq L(\mathcal{A})$ and $L_2 \cap L(\mathcal{A}) = \emptyset$. We will use the fact that Cartesian product $S_1 \times S_2$, ordered pointwise, is a wqo. The pointwise ordering on $S_1 \times S_2$ we denote by \preceq below.

For a word $w \in \Sigma^*$, we denote by $\delta(w)$ the pair of configuration $(s_1, s_2) \in S_1 \times S_2$, where s_1 is the unique configuration such that $i \xrightarrow{w} s_1$ in \mathcal{W}_1 , and similarly s_2 is the unique configuration such that $i \xrightarrow{w} s_2$ in \mathcal{W}_2 . A word $w \in \Sigma^*$ we call a *barrier* if for some strict prefix $u \in \Sigma^*$ of w , we have

$$\delta(u) \preceq \delta(w). \tag{1}$$

As states Q of the NFA \mathcal{A} we take all words $w \in \Sigma^*$ such that no strict prefix of w is a barrier (while the word w itself may be a barrier). We claim that the set Q is finite. To see this, consider an infinite directed tree, whose nodes are all finite words over Σ , and whose edges relate every word w with every word wa , for every $a \in \Sigma$. In particular, the empty word ε is the root. As $(S_1 \times S_2, \preceq)$ is a wqo, on every path of the tree there is a pair of nodes u, w satisfying (1), where u is a strict prefix of w . In other words, every path contains a barrier. Cut out all *strict* descendants of every barrier in the tree – this yields a tree t whose nodes are exactly the states Q . As the branching in t is finite, and it contains no infinite paths, by König's Lemma³ it is finite.

Now we define the transitions of \mathcal{A} . First, every edge (w, wa) in t determines a transition $(w, a, wa) \in T$ in \mathcal{A} . Second, we add ε -transitions (w, ε, u) for every barrier w and a strict prefix u of w satisfying (1); note that a word w may admit many such prefixes.⁴ The initial state is the root ε of t . Finally, accepting states are those words in Q that belong to L_1 , i.e. $F = Q \cap L_1$.

Lemma 10. *For every word $w \in \Sigma^*$ and state $q \in Q$ such that some run of \mathcal{A} over w finishes in q ,*

1. $w \in L_1$ implies $q \in L_1$; and
2. $w \in L_2$ implies $q \in L_2$.

Proof. As the sets F_1 and F_2 of final configurations are downward closed, we deduce the lemma directly from the following claim:

Claim. For every word $w \in \Sigma^*$ and state $q \in Q$ such that some run of \mathcal{A} over w finishes in q ,

$$\delta(q) \preceq \delta(w). \quad (2)$$

The claim is proved by an easy induction over the length of a run. In case of the empty run, w and q (the initial state of \mathcal{A}) are both the empty words, and therefore (2) holds as equality. In the induction step we assume $\delta(q) \preceq \delta(w)$ and consider a transition of \mathcal{A} from q . In case of an ε -transition $q \xrightarrow{\varepsilon} q'$ of \mathcal{A} we recall that the construction of \mathcal{A} guarantees that $\delta(q') \preceq \delta(q)$, which yields (by transitivity of \preceq) $\delta(q') \preceq \delta(w)$, as required. In case of a visible transition $q \xrightarrow{a} qa$ of \mathcal{A} , we put:

$$\begin{aligned} \delta(w) &= (s_1, s_2) & \delta(wa) &= (r_1, r_2) \\ \delta(q) &= (s'_1, s'_2) & \delta(qa) &= (r'_1, r'_2). \end{aligned}$$

We know that $\delta(q) \preceq \delta(w)$ which implies $s'_1 \preceq_1 s_1$; moreover $s_1 \xrightarrow{a} r_1$ and $s'_1 \xrightarrow{a} r'_1$ are the only a -labeled transitions of s_1 and s'_1 , respectively, in \mathcal{W}_1 . Therefore by downward compatibility we have $r'_1 \preceq_1 r_1$. Similarly we deduce $r'_2 \preceq r_2$, and in consequence $\delta(qa) \preceq \delta(wa)$, as required.

³ Recall that König's Lemma [19] says that every finitely branching infinite tree has an infinite path.

⁴ Equivalently, we could leave only one of the ε -transitions in every barrier state.

With Lemma 10 we easily show that $R = L(\mathcal{A})$ separates L_1 and L_2 . In order to see that $L_1 \subseteq R$, observe that Lemma 10 guarantees that every run of \mathcal{A} over every word from L_1 ends in an accepting state. For proving disjointness of R and L_2 , observe that every run of \mathcal{A} over a word from L_2 ends in a state that (being a word) belongs to L_2 . If the ending state were accepting, i.e. belonged to L_1 , we would obtain nonempty intersection of L_1 and L_2 , contradictory with our assumptions. Theorem 3 is thus proved.

5 Remarks

We have reported on two observations concerning UWSTS languages. First, every language such that both the language itself, as well as its complement, are recognized by a finitely branching upward compatible WSTS, is necessarily a regular language. Second, every two disjoint languages of upward compatible WSTSes, quasi ordered by ω^2 -wqos, are separable by a regular language. The latter result follows from our third result that applies to DWSTSes: every two disjoint languages of deterministic downward compatible WSTSes are separable by a regular language.

It is not clear whether the assumptions of Theorems 1–3 are really necessary. In particular, we were able neither to drop the ω^2 -wqo assumption in Theorem 2, nor to provide a counterexample, i.e., a pair of UWSTSes, ordered by wqos but not by ω^2 -wqos, whose languages are disjoint but not regular separable. Moreover, we do not know whether the finite branching assumption is really necessary for proving Theorem 1; we conjecture that this is not the case, i.e., that every doubly UWSTS-language is regular.

Concerning possible continuation of the presented work, we see a plenty of possible generalizations. For instance, one can consider languages of trees instead of languages of words; or one can consider *well behaved transition systems* (WBTS in short) of [2] instead of WSTS. WBTS is a generalization of WSTS where the wqo assumption on underlying quasi order is relaxed to *finite antichains condition* (FAC in short): the quasi order admits only finite antichains. In other words, one drops the requirement that a quasi order is well-founded. Our proofs do not work for WBTSes. For instance, in the proof of Theorem 1 we only need the pairwise ordering \subseteq_2 to satisfy FAC; we can not however relax the assumption on the given quasi orders \preceq_1 and \preceq_2 to FAC, as FAC is not even preserved by Cartesian product.

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